

## Advanced self-similar solutions of regular and irregular diffusion equations

E. Kovács<sup>1\*</sup>, L. Mátyás<sup>2</sup>, I.F. Barna<sup>3</sup>, S. Mahmoud<sup>1</sup>

<sup>1</sup>Institute of Physics and Electrical Engineering, University of Miskolc, 3515 Miskolc, Hungary

<sup>2</sup>Dept. of Bioengineering, Sapientia Hungarian U. of Transylvania Libertatii sq. 1, 530104 Miercurea Ciuc, Romania

<sup>3</sup>Wigner Research Center for Physics, Konkoly-Thege Miklós út 29 - 33, 1121 Budapest, Hungary

\*kendre01@gmail.com

We study various diffusion equations with an appropriate transformation of variables and with the reduction technique. With the self-similar and related Ansätze, we reduce the PDE to an ordinary differential equation (ODE). The derived solutions of the PDE belong to a family of functions which are presented for the case of the infinite horizon. This study is the second in the line of our investigation. In our first publication, we concentrated on the regular diffusion [1] and presented numerous solutions which are far beyond the well-known Gaussian or error type functions. Now we supplement the analysis in the above mentioned paper with additional generalized self-similar trial functions. We also consider time-dependent diffusion coefficients as well [4]. All of the obtained analytic solutions can be expressed with the Kummer- or Whittaker-type of functions. These kinds of investigations are organically linked to our long-term scientific activity in which we systematically examine fundamental hydrodynamic systems [2,3] and analyze physically relevant self-similar and traveling wave solutions. We are also developing numerical methods which are explicit and unconditionally stable at the same time, at least for the linear diffusion equation, see e.g. [4, 5] and the references therein.

After the following change of variables

$$C = \frac{1}{t^\alpha} f\left(\frac{x}{t^{1/2}}\right). \quad (1)$$

the equation of diffusion is transformed as follows

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \rightarrow -\alpha f - \frac{1}{2} \eta \frac{\partial f}{\partial \eta} = D \frac{\partial^2 f}{\partial \eta^2} \quad (2)$$

where  $\eta = x/\sqrt{t}$ . The solution of this ODE can be given by the Kummer functions:

$$f(\eta) = \eta e^{-\frac{\eta^2}{4D}} \left[ c_1 M\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right) \right] \quad (3)$$

The most relevant solutions for positive integer values of  $\alpha$  can be seen in Fig. 1.

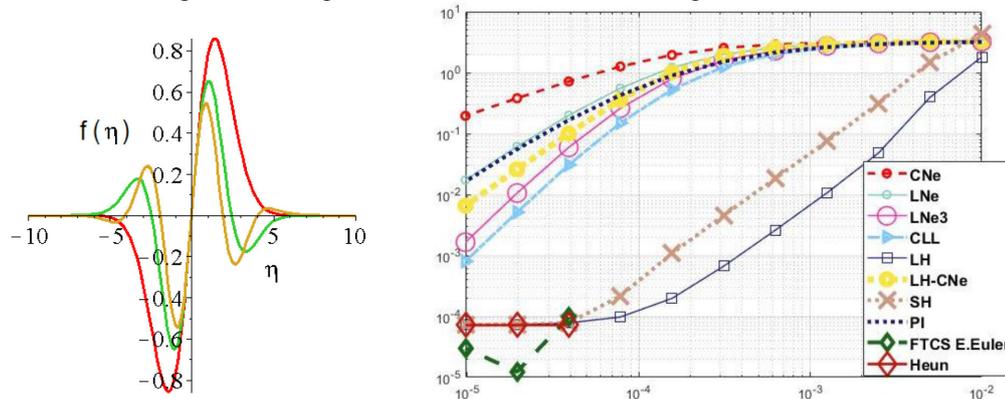


Figure 1: Left: Solutions of the ODE in (3), for  $\alpha=1$  (red),  $\alpha=2$  (green), and  $\alpha=3$  (yellow). Right: Maximum errors as a function of the time step size for 10 different numerical methods.

We transformed back this solution to obtain the solution of the original PDE. Then we reproduced that solution by 8 of our recent methods and by two standard (conditionally stable) methods. On the right side of Fig. 1 one can see the maximum ( $L_\infty$ ) errors as a function of the time step size for  $\alpha=4$ . The curves with circle markers represent methods which follows the maximum-minimum principle, which is a much stronger property than unconditional stability.

## References

- [1] L. Mátyás, I.F. Barna: *General self-similar solutions of diffusion equation and related constructions*, Romanian Journal of Physics 67, 101-117 (2022).
- [2] I.F. Barna, L. Mátyás: *Analytic solutions for the three dimensional compressible Navier-Stokes equation* Fluid Dynamic Research 46, 055508 (2014).
- [3] I.F. Barna, L. Mátyás, *Analytic self-similar solutions of the Oberbeck-Boussinesq equation*, Chaos Solitons and Fractals 78, 249 - 255 (2015).
- [4] Á Nagy, I. Omle; H. Kareem; E. Kovács; I.F. Barna; G. Bogнар, Stable, Explicit, Leapfrog-Hopscotch Algorithms for the Diffusion Equation. *Computation* 2021, **9**, 92. <https://doi.org/10.3390/computation9080092>
- [5] E. Kovács; Á Nagy, M. Saleh, A New Stable, Explicit, Third-Order Method for Diffusion-Type Problems, *Advanced Theory and Simulations*, 2022, accepted for publication